The algebra of Kleene stars of the plane and polylogarithms.

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ABSTRACT

We extend the definition and study the algebraic properties of the polylogarithm Li_T, where T is rational series over the alphabet $X=\{x_0,x_1\}$ belonging to $(\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$.

Keywords

Algebraically independent ; Polylogarithms ; Transcendent.

1. Introduction

In all the sequel of this text,

1. We consider the differential forms
$$\omega_0(z)=\frac{dz}{z} \quad\text{and}\quad \omega_1(z)=\frac{dz}{1-z}.$$
We denote Ω the cleft plane $\mathbb{C}-(]-\infty,0]\cup[1,+\infty[)$ and λ the rational fraction $z(1-z)^{-1}$ belonging to the differential unitary ring $\mathscr{C}:=\mathbb{C}[z,z^{-1},(1-z)^{-1}]$ with the differential operator $\partial_z:=d/dz$ and with the unitary element
$$\begin{aligned}1_\Omega:\Omega &\longrightarrow \mathbb{C},\\ z &\longmapsto 1.\end{aligned}$$

2. We construct, over the alphabets
$$X=\{x_0,x_1\},\quad Y=\{y_k\}_{k\ge1}\quad\text{and}\quad Y_0=Y\cup\{y_0\},$$
totally ordered by $x_0<x_1$ and $y_0>y_1>\cdots$ respectively, the bialgebras [1]
$$\begin{aligned}&(\mathbb{C}\langle X\rangle,\mathtt{conc},\Delta_{\sqcup\!\sqcup},1_{X^*},\varepsilon),\\ &(\mathbb{C}\langle Y\rangle,\mathtt{conc},\Delta_{\text{⧢}-},1_{Y^*},\varepsilon),\\ &(\mathbb{C}\langle Y_0\rangle,\mathtt{conc},\Delta_{\text{⧢}-},1_{Y_0^*},\varepsilon).\end{aligned}$$
These algebras, when endowed with their dual laws, are equipped with pure transcendence bases in bijection with the set of Lyndon words $\mathscr{L}yn(X)$, $\mathscr{L}yn(Y)$ and $\mathscr{L}yn(Y_0)$ respectively.

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1. Which are all Hopf save the last one due to y_0 which is infiltration-like [2].

Let us consider also the following morphism
$$\begin{aligned}\pi_Y^\circ:(\mathbb{C}\oplus\mathbb{C}\langle X\rangle x_1,\mathtt{conc}) &\longrightarrow (\mathbb{C}\langle Y\rangle,.),\\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 &\longmapsto y_{s_1}\ldots y_{s_r},\end{aligned}$$
for $r\ge1$ and, for any $a\in\mathbb{C},\pi_Y^\circ(a)=a$. The extension of π_Y° over $\mathbb{C}\langle X\rangle$ is denoted by $\pi_Y:\mathbb{C}\langle X\rangle\longrightarrow\mathbb{C}\langle Y\rangle$ satisfying, for any $p\in\mathbb{C}\langle X\rangle x_0$, $\pi_Y(p)=0$. Hence,
$$\ker(\pi_Y)=\mathbb{C}\langle X\rangle x_0 \quad\text{and}\quad \mathrm{Im}\,(\pi_Y)=\mathbb{C}\langle Y\rangle.$$
Let π_X be the inverse of π_Y° :
$$\begin{aligned}\pi_X:\mathbb{C}\langle Y\rangle &\longrightarrow \mathbb{C}\oplus\mathbb{C}\langle X\rangle x_1,\\ y_{s_1}\ldots y_{s_r} &\longmapsto x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1.\end{aligned}$$
The projectors π_X [2] and π_Y° are mutual adjoints :
$$\forall p\in\mathbb{C}\langle X\rangle,\forall q\in\mathbb{C}\langle Y\rangle,\qquad \langle\pi_Y(p)\mid q\rangle=\langle p\mid\pi_X(q)\rangle.$$

In continuation of [5, 7], the principal object of the present work is the *polylogarithm* well defined, for any r-uplet $(s_1,\ldots,s_r)\in\mathbb{C}^r$, $r\in\mathbb{N}_+$ and for any $z\in\mathbb{C}$ such that $|z|<1$, as follows
$$\mathrm{Li}_{s_1,\ldots,s_r}(z):=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{1}$$
Then the Taylor expansion of the function $(1-z)^{-1}\mathrm{Li}_{s_1,\ldots,s_r}(z)$ is given by
$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z}=\sum_{N\ge0}\mathrm{H}_{s_1,\ldots,s_r}(N)\,z^N,$$
where the coefficient $\mathrm{H}_{s_1,\ldots,s_r}:\mathbb{N}\longrightarrow\mathbb{Q}$ is an arithmetic function, also called *harmonic sum*, which can be expressed as follows
$$\mathrm{H}_{s_1,\ldots,s_r}(N):=\sum_{N\ge n_1>\ldots>n_r>0}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{2}$$

From the analytic continuation of polyzetas [9, 24], for any $r\ge1$, if $(s_1,\ldots,s_r)\in\mathscr{H}_r$ satisfies (3) then [3], after a theorem by Abel, one obtains the *polyzeta* as follows
$$\lim_{z\to1}\mathrm{Li}_{s_1,\ldots,s_r}(z)=\lim_{N\to\infty}\mathrm{H}_{s_1,\ldots,s_r}(N)=\zeta(s_1,\ldots,s_r).$$
This theorem is no more valid in the divergent cases (for $(s_1,\ldots,s_r)\in\mathbb{N}^r$) and require the renormalization of the corresponding divergent

2. With a little abuse of language, π_X is now considered as targeted to $\mathbb{C}\langle X\rangle$.

3. For $r\ge1$, $\zeta(s_1,\ldots,s_r)$ is as a meromorphic function on
$$\mathscr{H}_r=\{(s_1,\ldots,s_r)\in\mathbb{C}^r|\forall m=1,\ldots,r,\Re(s_1)+\ldots+\Re(s_m)>1\}. \tag{3}$$

polyzetas. It is already done for the corresponding case of polyzetas at positive multi-indices [3, 4, 20] and it is done [8, 11, 22] and completed in [5, 7] for the case of polyzetas at positive multi-indices.

To study the polylogarithms at negative multi-indices, one relies on [5, 7]

1. the (one-to-one) correspondence between the multi-indices $(s_1,\ldots,s_r)\in\mathbb{N}^r$ and the words $y_{s_1}\ldots y_{s_r}$ defined over Y_0,
2. indexing these polylogarithms by words $y_{s_1}\ldots y_{s_r}$:

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}(z)=\mathrm{Li}^-_{s_1,\ldots,s_r}(z)=\sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}\,z^{n_1}.$$

In the same way, for polylogarithms at positive indices, one relies on [15, 17]

1. the (one-to-one) correspondence between the combinatorial compositions $(s_1,\ldots,s_r)$ and the words $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ in $X^*x_1+1_{X^*}$
2. the indexing of these polylogarithms by words $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$:

$$\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(z)=\mathrm{Li}_{s_1,\ldots,s_r}(z)=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}.$$

Moreover, one obtained the polylogarithms at positive indices as image by the following isomorphism of the shuffle algebra [4] [15]

$$\begin{array}{rcl}\mathrm{Li}_\bullet:(\mathbb{C}\langle X\rangle,ш,1_{X^*}) & \longrightarrow & (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1_\Omega),\\ x_0^n & \longmapsto & \dfrac{\log^n(z)}{n!},\\ x_1^n & \longmapsto & \dfrac{\log^n(1/(1-z))}{n!},\\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \longmapsto & \displaystyle\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}.\end{array}$$

Extending over the set of rational power series [5] on non commutative variables, $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, as follows

$$\begin{aligned}S &= \sum_{n\ge0}\langle S\mid x_0^n\rangle x_0^n+\sum_{k\ge1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S\mid w\rangle\,w,\\ \mathrm{Li}_S(z) &= \sum_{n\ge0}\langle S\mid x_0^n\rangle\frac{\log^n(z)}{n!}+\sum_{k\ge1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S\mid w\rangle\,\mathrm{Li}_w,\end{aligned}$$

the morphism $\mathrm{Li}_\bullet$ is no longer injective over $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ but $\{\mathrm{Li}_w\}_{w\in X^*}$ are still linearly independant over $\mathscr{C}$ [20, 19].

EXAMPLE 1. *i.* $1_\Omega=\mathrm{Li}_{1_{X^*}}=\mathrm{Li}_{x_1^*-x_0^*ш x_1^*}$.
ii. $\lambda=\mathrm{Li}_{(x_0+x_1)^*}=\mathrm{Li}_{x_0^*ш x_1^*}=\mathrm{Li}_{x_1^*-1}$.
iii. $\mathscr{C}=\mathbb{C}[\mathrm{Li}_{x_0^*},\mathrm{Li}_{(-x_0)^*},\mathrm{Li}_{x_1^*}]$.
iv. $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}=\{\mathrm{Li}_S\,|S\in\mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*]ш\mathbb{C}\langle X\rangle\}$.

Let us consider also the differential and integration operators, acting on $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ [21] :

$$\partial_z=\frac{d}{dz},\theta_0=z\frac{d}{dz},\theta_1=(1-z)\frac{d}{dz},$$

$$\forall f\in\mathscr{C},\qquad \iota_0(f)=\int_{z_0}^z f(s)\omega_0(s)\quad\text{and}\quad \iota_1(f)=\int_0^z f(s)\omega_1(s).$$

4. As follows defined on a superset of the of Lyndon words, as pure transcendence basis, and extended by algebraic specialization [12, 13].

5. $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ is the closure by rational operations $\{+,\mathtt{conc},^*\}$ of $\mathbb{C}\langle X\rangle$, where, for $S\in\mathbb{C}\langle\langle X\rangle\rangle$ such that $\langle S\mid 1_{X^*}\rangle=0$, one has [1]

$$S^*=\sum_{k\ge0}S^k.$$

Here, the operator ι_0 is well-defined (as in definition 1 in section 2.2) then one can check easily [18, 19, 5, 7]

1. The subspace $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the action of $\{\theta_0,\theta_1\}$ and $\{\iota_0,\iota_1\}$.
2. The operators $\{\theta_0,\theta_1,\iota_0,\iota_1\}$ satisfy in particular,

$$\begin{aligned}\theta_1+\theta_0=[\theta_1,\theta_0]=\partial_z \quad&\text{and}\quad \forall k=0,1,\theta_k\iota_k=\mathrm{Id},\\ [\theta_0\iota_1,\theta_1\iota_0]=0\quad&\text{and}\quad(\theta_0\iota_1)(\theta_1\iota_0)=(\theta_1\iota_0)(\theta_0\iota_1)=\mathrm{Id}.\end{aligned}$$

3. $\theta_0\iota_1$ and $\theta_1\iota_0$ are scalar operators within $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$, respectively with eigenvalues λ and $1/\lambda$, *i.e.*

$$(\theta_0\iota_1)f=\lambda f\quad\text{and}\quad(\theta_1\iota_0)f=(1/\lambda)f.$$

4. Let $w=y_{s_1}\ldots y_{s_r}\in Y^*$ (then $\pi_X(w)=x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$) and $u=y_{t_1}\ldots y_{t_r}\in Y_0^*$. The functions $\mathrm{Li}_w,\mathrm{Li}^-_u$ satisfy

$$\begin{aligned}\mathrm{Li}_w=(\iota_0^{s_1-1}\iota_1\ldots\iota_0^{s_r-1}\iota_1)1_\Omega,\qquad&\mathrm{Li}^-_u=(\theta_0^{t_1+1}\iota_1\ldots\theta_0^{t_r+1}\iota_1)1_\Omega,\\ \iota_0\,\mathrm{Li}_{\pi_X(w)}=\mathrm{Li}_{x_0\pi_X(w)},\qquad&\iota_1\,\mathrm{Li}_w=\mathrm{Li}_{x_1\pi_X(w)},\\ \theta_0\,\mathrm{Li}_{x_0\pi_X(w)}=\mathrm{Li}_{\pi_X(w)},\qquad&\theta_1\,\mathrm{Li}_{x_1\pi_X(w)}=\mathrm{Li}_{\pi_X(w)},\\ \theta_0\,\mathrm{Li}_{x_1\pi_X(w)}=\lambda\,\mathrm{Li}_{\pi_X(w)},\qquad&\theta_0\,\mathrm{Li}_{x_1\pi_X(w)}=\mathrm{Li}_{\pi_X(w)}/\lambda.\end{aligned}$$

Here, we explain the whole project of extension of $\mathrm{Li}_\bullet$, study different aspects of it, in particular what is desired of $\iota_i,i=0,1$. The interesting problem in here is that what we do expect of $\iota_i,i=0,1$. In fact, the answers are

— it is a section of $\theta_i,i=0,1$ (*i.e.* takes primitives for the corresponding differential operators).

— it extends $\iota_i,i=0,1$ (defined on $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$, and very surprisingly, although not coming directly from Chen calculus, they provide a group-like generating series)

We will use this construction to extend $\mathrm{Li}_\bullet$ to $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and, after that, we extend it to a much larger rational algebra.

2. Background

2.1 Standard topology on $\mathscr{H}(\Omega)$

The algebra $\mathscr{H}(\Omega)$ is that of analytic functions defined over Ω. It is endowed with the topology of *compact convergence* whose seminorms are indexed by compact subsets of Ω, and defined by

$$p_K(f)=||f||_K=\sup_{s\in K}|f(s)|.$$

Of course,

$$p_{K_1\cup K_2}=\sup(p_{K_1},p_{K_2}),$$

and therefore the same topology is defined by extracting a *fondamental subset of seminorms*, here it can be choosen denumerable. As $\mathscr{H}(\Omega)$ is complete with this topology it is a Frechet space and even, as

$$p_K(fg)\le p_K(f)p_K(g),$$

it is a Frechet algebra (even more, as $p_K(1_\Omega)=1$ a Frechet algebra with unit).

With the standard topology above, an operator $\phi\in\mathrm{End}(\mathscr{H}(\Omega))$ is continuous iff (with K_i compacts of Ω)

$$(\forall K_2)(\exists K_1)(\exists M_{12}>0)(\forall f\in\mathscr{H}(\Omega))(||\phi(f)||_{K_2}\le M_{12}||f||_{K_1}).$$

2.2 Study of continuity of the sections θ_i and ι_i

Then, $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (and $\mathscr{H}(\Omega)$) being closed under the operators $\theta_i;\ i=0,1$. We will first build sections of them, then see that they are continuous and, propose (discontinuous) sections more adapted to renormalisation and the computation of associators.

For $z_0 \in \Omega$, let us define $\iota_i^{z_0} \in \mathrm{End}(\mathscr{H}(\Omega))$ by

$$\iota_0^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_0(s), \qquad \iota_1^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_1(s).$$

It is easy to check that $\theta_i \iota_i^{z_0} = Id_{\mathscr{H}(\Omega)}$ and that they are continuous on $\mathscr{H}(\Omega)$ for the topology of compact convergence because for all $K \subset_{compact} \Omega$, we have

$$|p_K(\iota_i^{z_0}(f)| \leq p_K(f)\Big[\sup_{z\in K}|\int_{z_0}^{z}\omega_i(s)|\Big],$$

and, in view or paragraph (2.1), this is sufficient to prove continuity. Hence the operators $\iota_i^{z_0}$ are also well defined on $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and it is easy to check that

$$\iota_i^{z_0}(\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}) \subset \mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

Due to the decomposition of $\mathscr{H}(\Omega)$ into a direct sum of closed subspaces

$$\mathscr{H}(\Omega) = \mathscr{H}_{z_0\mapsto 0}(\Omega) \oplus \mathbb{C}1_\Omega,$$

it is not hard to see that the graphs of θ_i are closed, thus, the θ_i are also continuous.

Much more interesting (and adapted to the explicit computation of associators) we have the operator ι_i (without superscripts), mentioned in the introduction and (rigorously) defined by means of a $\mathbb{C}$-basis of

$$\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*} = \mathscr{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

As $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \cong \mathbb{C}[\mathscr{L}yn(X)]$, one can partition the alphabet of this polynomial algebra in $(\mathscr{L}yn(X)\cap X^*x_1)\sqcup\{x_0\}$ and then get the decomposition

$$\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*} = \mathscr{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*x_1} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in x_0^*}.$$

In fact, we have an algorithm to convert $\mathrm{Li}_{ux_1x_0^n}$ as

$$\mathrm{Li}_{ux_1x_0^n}(z) = \sum_{m\leq n} P_m \log^m(z) = \sum_{\substack{m\leq n\\ w\in X^*x_1}} \langle P_m \mid w\rangle \mathrm{Li}_w(z)\log^m(z).$$

due to the identity [13]

$$ux_1x_0^n = ux_1 \sqcup\!\sqcup\, x_0^n - \sum_{k=1}^{n} (u \sqcup\!\sqcup\, x_0^k)x_1x_0^{n-k}.$$

This means that

$$\begin{aligned}
\mathscr{B} &= \Big(z^k \mathrm{Li}_{ux_1}(z)\mathrm{Li}_{x_0^n}(z)\Big)_{(k,n,u)\in\mathbb{Z}\times\mathbb{N}\times X^*}\\
&\sqcup \Big(\frac{1}{(1-z)^l}\mathrm{Li}_{ux_1}(z)\mathrm{Li}_{x_0^n}(z)\Big)_{(l,n,u)\in\mathbb{N}_+^2\times X^*}\\
&\sqcup \Big(z^k \mathrm{Li}_{x_0^n}(z)\Big)_{(k,n)\in\mathbb{Z}\times\mathbb{N}_+}\\
&\sqcup \Big(\frac{1}{(1-z)^l}\mathrm{Li}_{x_0^n}(z)\Big)_{(k,l,n)\in\mathbb{N}_+^2}
\end{aligned}$$

is a $\mathbb{C}$-basis of $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$.

With this basis, we can define the operator ι_0 as follows

DEFINITION 1. *Define the index map* $\mathrm{ind} : \mathscr{B} \to \mathbb{Z}$ *by*

$$\begin{aligned}
\mathrm{ind}\Big(\frac{z^k}{(1-z)^l}\mathrm{Li}_{x_0^n}(z)\Big) &= k,\\
\mathrm{ind}\Big(\frac{z^k}{(1-z)^l}\mathrm{Li}_{ux_1}(z)\log^n(z)\Big) &= k+|ux_1|.
\end{aligned}$$

Now ι_0 is computed by :

1. $\iota_0(b) = \int_0^z b(s)\omega_0(s)$ *if* $\mathrm{ind}(b) \geq 1$.
2. $\iota_0(b) = \int_1^z b(s)\omega_0(s)$ *if* $\mathrm{ind}(b) \leq 0$.

To show discontinuity of ι_0 with a direct example, the technique consists in exhibiting 2 sequences $f_n, g_n \in \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ converging to the same limit but such that

$$\lim \iota_0(f_n) \neq \lim \iota_0(g_n).$$

Here we choose the function z which can be approached by two limits. For continuity, we should have "equality of the limits of the image-sequences" which is not the case.

$$\begin{aligned}
z &= \sum_{n\geq 0} \frac{\log^n(z)}{n!},\\
z &= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n!}\log^n(\frac{1}{1-z}).
\end{aligned}$$

Let then

$$f_n = \sum_{0\leq m\leq n}\frac{\log^m(z)}{m!} \quad \text{and} \quad g_n = \sum_{1\leq m\leq n}\frac{(-1)^{m+1}}{m!}\log^m(\frac{1}{1-z}),$$

we have $f_n, g_n \in \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ and $\iota_0(f_n) = f_{n+1}-1$. Hence, one has $\lim(\iota_0(f_n)) = z-1$. On the other hand

$$\lim \iota_0(g_n) = \lim \int_0^z g_n(s)\omega_0(s) = \int_0^z \lim g_n(s)\omega_0(s) = \int_0^z s\omega_0(s) = z.$$

The exchange of the integral with the limit above comes from the fact that the operator

$$\phi \mapsto \int_0^z \phi(s)\omega_0(s),$$

is continuous on the space $\mathscr{H}_0(\Omega\cup B(0,1))$ of analytic functions $f \in \mathscr{H}(\Omega\cup B(0,1))$ such that $f(0)=0$ ($B(0,1)$ is the open ball of center 0 and radius 1).

3. Algebraic extension of $\mathrm{Li}_\bullet$ to $(\mathbb{C}^{\mathrm{rat}}\langle\langle \mathbf{X}\rangle\rangle, \sqcup\!\sqcup, \mathbf{1}_{\mathbf{X}^*})[\mathbf{x_0^*}, (-\mathbf{x_0})^*, \mathbf{x_1^*}]$

We will use several times the following lemma which is characteristic-free.

LEMMA 1. *Let $(\mathscr{A}, d)$ be a commutative differential ring without zero divisor, and $R = \ker(d)$ be its subring of constants. Let $z \in \mathscr{A}$ such that $d(z) = 1$ and $S = \{e_\alpha\}_{\alpha\in I}$ be a set of eigenfunctions of d all different ($I \subset R$) i.e.*

i. $e_\alpha \neq 0$.

ii. $d(e_\alpha) = \alpha e_\alpha$; $\alpha \in I$.

Then the family $(e_\alpha)_{\alpha\in I}$ is linearly free over $R[z]$ [6].

PROOF. If there is no non-trivial $R[z]$-linear relation, we are done. Otherwise let us consider relations

$$\sum_{j=1}^{N} P_j(z)e_{\alpha_j} = 0, \tag{4}$$

with $P_j \in R[t]_{pol}\setminus\{0\}$ [7] for all j (packed linear relations). We choose one minimal w.r.t. the triplet

$$[N, \deg(P_N), \sum_{j<N}\deg(P_j)], \tag{5}$$

6. Here $R[z]$ is understood as ring adjunction i.e. the smallest subring generated by $R\cup\{z\}$.

7. Here $R[t]_{pol}$ means the formal univariate polynomial ring (the subscript is here to avoid confusion).

lexicographically ordered from left to right [8].

Remarking that $d(P(z)) = P'(z)$ (because $d(z) = 1$), we apply the operator $d - \alpha_N Id$ to both sides of (4) and get

$$\sum_{j=1}^{N} \Big(P'_j(z) + (\alpha_j - \alpha_N)P_j(z)\Big)e_{\alpha_j} = 0. \tag{6}$$

Minimality of (4) implies that (6) is trivial i.e.

$$P'_N(z) = 0\ ;\ (\forall j = 1..N-1)(P'_j(z) + (\alpha_j - \alpha_N)P_j(z) = 0). \tag{7}$$

Now relation (4) implies

$$\prod_{1\le j\le n-1} (\alpha_N - \alpha_j)\Big(\sum_{j=1}^{N} P_j(z)e_{\alpha_j}\Big) = 0, \tag{8}$$

which, because $\mathscr{A}$ has no zero divisor, is packed and has the same associated triplet (5) as (4). From (7), we see that for all $k < N$

$$\prod_{1\le j\le n-1} (\alpha_N - \alpha_j)P_k(z) = \prod_{\substack{1\le j\le n-1\\ j\neq k}} (\alpha_N - \alpha_j)P'_k(z),$$

so, if $N \geq 2$, we would get a relation of lower triplet (5). This being impossible, we get $N = 1$ and (4) boils down to $P_N(z)e_N = 0$ which, as $\mathscr{A}$ has no zero divisor, implies $P_N(z) = 0$, contradiction.

Then the $(e_\alpha)_{\alpha\in I}$ are $R[z]$-linearly independent. $\square$

REMARK 1. *If $\mathscr{A}$ is a $\mathbb{Q}$-algebra or only of characteristic zero (i.e., $n1_{\mathscr{A}} = 0 \Rightarrow n = 0$), then $d(z) = 1$ implies that z is transcendent over R.*

First of all, let us prove

LEMMA 2. *Let k be a field of characteristic zero and Z an alphabet. Then $(k\langle\langle Z\rangle\rangle, \sqcup\!\sqcup, 1_{Z^*})$ is a k-algebra without zero divisor.*

PROOF. Let $B = (b_i)_{i\in I}$ be an ordered basis of $\mathscr{L}ie_k\langle Z\rangle$ and $(\frac{B^\alpha}{\alpha!})_{\alpha\in\mathbb{N}^{(I)}}$ the divided corresponding PBW basis. One has

$$\Delta_{\sqcup\!\sqcup}(\frac{B^\alpha}{\alpha!}) = \sum_{\alpha=\alpha_1+\alpha_2} \frac{B^{\alpha_1}}{\alpha_1!} \otimes \frac{B^{\alpha_2}}{\alpha_2!}\ .$$

Hence, if $S, T \in (k\langle\langle Z\rangle\rangle, \sqcup\!\sqcup, 1_{Z^*})$, considering

$$\langle S \sqcup\!\sqcup T \mid \frac{B^\alpha}{\alpha!}\rangle = \langle S\otimes T \mid \Delta_{\sqcup\!\sqcup}(\frac{B^\alpha}{\alpha!})\rangle = \sum_{\alpha=\alpha_1+\alpha_2} \langle S \mid \frac{B^{\alpha_1}}{\alpha_1!}\rangle\langle T \mid \frac{B^{\alpha_2}}{\alpha_2!}\rangle\ ,$$

we see that $(k\langle\langle Z\rangle\rangle, \sqcup\!\sqcup, 1_{Z^*}) \simeq (k[[I]], ., 1)$ (commutative algebra of formal series) which has no zero divisor). $\square$

LEMMA 3. *Let $\mathscr{A}$ be a $\mathbb{Q}$-algebra (associative, unital, commutative) and z an indeterminate, then $e^z \in \mathscr{A}[[z]]$ is transcendent over $\mathscr{A}[z]$.*

PROOF. It is straightforward consequence of Remark (1). Note that this can be rephrased as $[z, e^z]$ are algebraically independant over $\mathscr{A}$. $\square$

PROPOSITION 1. *Let $Z = \{z_n\}_{n\in\mathbb{N}}$ be an alphabet, then $[e^{z_0}, e^{z_1}]$ is algebraically independent on $\mathbb{C}[Z]$ within $\mathbb{C}[[Z]]$.*

PROOF. Using lemma 3, one can prove by recurrence that

$$[e^{z_0}, e^{z_1}, \cdots, e^{z_k}, z_0, z_1, \cdots, z_k],$$

are algebraically independent. This implies that $Z \sqcup \{e^z\}_{z\in Z}$ is an algebraically independent set and, by restriction $Z \sqcup \{e^{z_0}, e^{z_1}\}$ whence the proposition. $\square$

8. i.e. consider the ones with N minimal and among these, we choose one with $\deg(P_N)$ minimal and among these we choose one with $\sum_{j<N} \deg(P_j)$ minimal.

COROLLARY 1. *i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, \sqcup\!\sqcup, 1_{X^*})$.*

ii. $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w \sqcup\!\sqcup (x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup (x_1^*)^{\sqcup\!\sqcup\, l}\}_{\substack{w\in X^*\\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

PROOF. Chase denominators and use a theorem by Radford [23] with $Z = \mathscr{L}yn(X)$. $\square$

COROLLARY 2. *There exists a unique morphism μ, from $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathscr{H}(\Omega)$ defined by*

i. $\mu(w) = \mathrm{Li}_w$ for any $w \in X^*$,

ii. $\mu(x_0^*) = z$,

iii. $\mu((-x_0)^*) = 1/z$,

iv. $\mu(x_1^*) = 1/(1-z)$.

DEFINITION 2. *We call $\mathrm{Li}_\bullet^{(1)}$ the morphism μ.*

Remark that the image of $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ by $\mathrm{Li}_\bullet^{(1)}$ (sect. 3) is exactly $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$. And the operator ι_0 defined by means of $\mathrm{Li}_\bullet$ is the same as the one defined by tensor decomposition. We have a diagram as follows

$$\begin{tikzcd}
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \arrow[r, hook, two heads, "\mathrm{Li}_\bullet"] \arrow[d, hook] & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \arrow[r, two heads, "\mathrm{Li}_\bullet^{(1)}"] \arrow[d, hook] & \mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*} \arrow[d, hook] \\
\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[r, "\mathrm{Li}_\bullet^{(2)}"] & \mathscr{H}(\Omega) \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \arrow[u] \arrow[ur, dashed] &
\end{tikzcd}$$

DIAGRAM 1. *Arrows and spaces obtained in this project (so far). Among horizontal arrows only $\mathrm{Li}_\bullet$ is into (and is an isomorphism) $\mathrm{Li}_\bullet^{(1)}$ and $\mathrm{Li}_\bullet^{(2)}$ are not into (for example, the image of the non-zero element $x_0^* \sqcup\!\sqcup x_1^* - x_1^* + 1$ is zero, see Example 1). The image of $\mathrm{Li}_\bullet^{(2)}$ is presumably*

$$\mathrm{Im}(SP_{\mathbb{C}}(X))\{\mathrm{Li}_w\}_{w\in X^*} \simeq \mathbb{C}\{z^\alpha(1-z)^\beta\}_{\alpha,\beta\in\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

4. Extension to $\mathbb{C}\langle \mathbf{X}\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle \mathbf{x_0}\rangle\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}\langle\langle \mathbf{x_1}\rangle\rangle$

4.1 Study of the shuffle algebra $SP_{\mathbb{C}}(X)$

Indeed, the set $(a_0x_0 + a_1x_1)^*_{a_0,a_1\in\mathbb{C}}$ is a shuffle monoid as

$$(a_0x_0 + a_1x_1)^* \sqcup\!\sqcup (b_0x_0 + b_1x_1)^* = ((a_0 + b_0)x_0 + (a_1 + b_1)x_1)^*.$$

As there are many relations between these elements (as a monoid it is isomorphic to $\mathbb{C}^2$, hence far from being free), we study here the vector space

$$SP_{\mathbb{C}}(X) = \mathrm{span}_{\mathbb{C}}\{(a_0x_0 + a_1x_1)^*\}_{a_0,a_1\in\mathbb{C}}.$$

It is a shuffle sub-algebra of $(\mathbb{C})^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup (\mathbb{C})^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ which will be called *star of the plane*. Note that it is also a shuffle sub-algebra of the algebra $(\mathbb{C}^{exch}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$ of exchangeable series. We can give the

DEFINITION 3. *A series is said exchangeable iff whenever two words have the same multidegree (here bidegree) then they have the same coefficient within it. Formally for all $u, v \in X^*$*

$$\Big((\forall x \in X)(|u|_x = |v|_x)\Big) \implies \langle S \mid u\rangle = \langle S \mid v\rangle.$$

On the other hand, for any $S \in \mathbb{C}\langle\langle X\rangle\rangle$, we can write

$$S = \sum_{n\geq 0} P_n,$$

where $P_n \in \mathbb{C}[X]$ such that $\deg P_n = n$ for every $n \geq 0$. Then S is called exchangeable iff P_n are symmetric by permutations of places for every $n \in \mathbb{N}$. If S is written as above then we can write

$$P_n = \sum_{i=0}^{n} \alpha_{n,i} x_0^i \sqcup\!\sqcup\, x_1^{n-i}.$$

DEFINITION 4. *Let $S \in \mathbb{C}\langle\langle X\rangle\rangle$ (resp. $\mathbb{C}\langle X\rangle$) and let $P \in \mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}\langle\langle X\rangle\rangle$). The left and right residual of S by P are respectively the formal power series $P\triangleleft S$ and $S\triangleright P$ in $\mathbb{C}\langle\langle X\rangle\rangle$ defined by*

$$\langle P\triangleleft S \mid w\rangle = \langle S \mid wP\rangle \quad (\textit{resp.} \quad \langle S\triangleright P \mid w\rangle = \langle S \mid Pw\rangle).$$

For any $S \in \mathbb{C}\langle\langle X\rangle\rangle$ (resp. $\mathbb{C}\langle X\rangle$) and $P,Q \in \mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}\langle\langle X\rangle\rangle$), we straightforwardly get

$$\begin{aligned} P\triangleleft(Q\triangleleft S) &= PQ\triangleleft S,\\ (S\triangleright P)\triangleright Q &= S\triangleright PQ,\\ (P\triangleleft S)\triangleright Q &= P\triangleleft(S\triangleright Q).\end{aligned}$$

In case $x,y \in X$ and $w \in X^*$, we get [9]

$$x\triangleleft(wy) = \delta_x^y w \quad \text{and} \quad xw\triangleright y = \delta_x^y w.$$

THEOREM 1. *Le $\delta \in \mathfrak{Der}(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$. Moreover, we suppose that δ is locally nilpotent [10]. Then the family $(t\delta)^n/n!$ is summable and its sum, denoted* $\exp(t\delta)$*, is is a one-parameter group of automorphisms of* $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

THEOREM 2. *Let L be a Lie series [11]. Let δ_L^r and δ_L^l be defined respectively by*

$$\delta_L^r(P) := P\triangleleft L \quad \textit{and} \quad \delta_L^l(P) := L\triangleright P.$$

Then δ_L^r and δ_L^l are locally nilpotent derivations of $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

Therefore, $\exp(t\delta_L^r)$ *and* $\exp(t\delta_L^l)$ *are one-parameter groups of* $Aut(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ *and*

$$\exp(t\delta_L^r)P = P\triangleleft\exp(tL) \quad \textit{and} \quad \exp(t\delta_L^l)P = \exp(tL)\triangleright P.$$

It is not hard to see that the algebra $\mathbb{C}\langle X\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ is closed by the shuffle derivations [12] $\delta_{x_0}^l, \delta_{x_1}^l$. In particular, on it, these derivations commute [13] with $\delta_{x_0}^r$ and $\delta_{x_1}^r$, respectively, *i.e.*, for any $x \in X$ and $S \in \mathbb{C}\langle X\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$, one has

$$\delta_x^l(S) = \delta_x^r(S).$$

Moreover, one has

$$(\alpha\delta_{x_0}^l + \beta\delta_{x_1}^l)[(a_0x_0+a_1x_1)^*] = (\alpha a_0+\beta a_1)[(a_0x_0+a_1x_1)^*],$$

from this we get that the family $\{(a_0x_0+a_1x_1)^*\}_{a_0,a_1\in\mathbb{C}}$ is linearly free over $\mathbb{C}$

$$SP_{\mathbb{C}}(X) = \bigoplus_{(a_0,a_1)\in\mathbb{C}} \mathbb{C}\{(a_0x_0+a_1x_1)^*\}.$$

We can get an arrow of $\mathrm{Li}_\bullet^{(2)}$ type $(SP_{\mathbb{C}}(X), \sqcup\!\sqcup, 1_{X^*}) \longrightarrow \mathscr{H}(\Omega)$ by sending

$$(a_0x_0+a_1x_1)^* = (a_0x_0)^* \sqcup\!\sqcup\, (a_1x_1)^* \quad \longmapsto \quad z^{a_0}(1-z)^{-a_1}.$$

In particular, for any $n \in \mathbb{N}_+$, one has

$$\mathrm{Li}^-_{\underbrace{0,\ldots,0}_{n \text{ times}}}(z) = \mathrm{Li}^{(2)}_{(nx_0+nx_1)^*}(z).$$

This arrow is a morphism for the shuffle product.

4.2 Study of the algebra $\mathbb{C}\langle\mathbf{X}\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle\mathbf{x_0}\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle\mathbf{x_1}\rangle\rangle$

We will start by the study of the one-letter shuffle algebra, *i.e.* $(\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle, \sqcup\!\sqcup, 1_{x^*})$ and use two times Lemma 1 above.

Let us now consider $\mathscr{A} = \mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle; e_\alpha = (\alpha x)^*, \alpha \in \mathbb{C}$ endowed with $d = \delta_x^l$ (which is a derivation for the shuffle) and $z = x$. We are in the conditions of Lemma 1 and then $((\alpha x)^*)_{\alpha\in\mathbb{C}}$ is $\mathbb{C}[x]$-linearly free which amounts to say that

$$B_0 = (x^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, (\alpha x)^*)_{k\in\mathbb{N},\alpha\in\mathbb{C}},$$

is $\mathbb{C}$-linearly free in $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle$.

To see that it is a basis, it suffices to prove that B_0 is (linearly) generating. Consider that

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P/Q\}_{P,Q\in\mathbb{C}[x],Q(0)\neq 0},$$

then, as $\mathbb{C}$ is algebraically closed, we have a basis

$$B_1 \cup B_2 = \{x^k\}_{k\geq 0} \cup \{((\alpha x)^*)^l\}_{\alpha\in\mathbb{C}^*,l\geq 1},$$

and it suffices to see that we can generate B_2 by elements of B_0, which s a consequence of the two identities

$$\begin{aligned} x\sqcup\!\sqcup\,((\alpha x)^*)^n &= \sum_{j=1}^{n+1} \alpha(n,j)((\alpha x)^*)^{\sqcup\!\sqcup\, j} \quad \text{with } \alpha(n,n+1)\neq 0,\\ x^k \sqcup\!\sqcup\, (\alpha x)^* &= \frac{1}{k!}(x^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, (\alpha x)^*).\end{aligned}$$

Now, we use again Lemma 1 with

$$\mathscr{A} = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \subset \mathbb{C}\langle\langle x_0,x_1\rangle\rangle,$$

hence without zero divisor (see Lemma 2), endowed with $d = \delta_{x_1}^l$ then $(x_0^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, (\alpha x_0)^*)_{\substack{k\in\mathbb{N};\\ \alpha\in\mathbb{C}}}$, is linearly free over $R = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$. It is easily seen, using a decomposition like

$$S = \sum_{p\geq 0,q\geq 0} \langle S \mid x_0^{\sqcup\!\sqcup\, p} \sqcup\!\sqcup\, x_1^{\sqcup\!\sqcup\, q}\rangle x_0^{\sqcup\!\sqcup\, p} \sqcup\!\sqcup\, x_1^{\sqcup\!\sqcup\, q},$$

that $\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle = \ker(d)$ and one obtains then that the arrow

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \to \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \subset \mathbb{C}^{\mathrm{rat}}\langle\langle x_0,x_1\rangle\rangle$$

is an isomorphism. Hence, $(x_0^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup\, (\alpha_0x_0)^* \sqcup\!\sqcup\, x_1^{\sqcup\!\sqcup\, k_1} \sqcup\!\sqcup\, (\alpha_1x_1)^*)_{\substack{k_i\in\mathbb{N};\\ \alpha_i\in\mathbb{C}}}$ is a $\mathbb{C}$-basis of $\mathscr{A} = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$. In order to extend $\mathrm{Li}_\bullet$ to $\mathscr{A}$ we send

$$T(k_0,k_1,\alpha_0,\alpha_1) = x_0^{\sqcup\!\sqcup\, k_0} \sqcup\!\sqcup\, (\alpha_0x_0)^* \sqcup\!\sqcup\, x_1^{\sqcup\!\sqcup\, k_1} \sqcup\!\sqcup\, (\alpha_1x_1)^*,$$

to $\log^{k_0}(z)z^{\alpha_0}\log^{k_1}(1/(1-z))(1/(1-z))^{\alpha_1}$, and see that the constructed arrow follows multiplication given by

$$T(j_0,j_1,\alpha_0,\alpha_1)T(k_0,k_1,\beta_0,\beta_1) = T(j_0+k_0,j_1+k_1,\alpha_0+\beta_0,\alpha_1+\beta_1).$$

9. For any words $u,v \in X^*$, if $u = v$ then $\delta_u^v = 1$ else 0.
10. $\phi \in \mathrm{End}(V)$ is said to be locally nilpotent iff, for any $v \in V$, there exists $N \in \mathbb{N}$ s.t. $\phi^N(v) = 0$.
11. *i.e.* $\Delta_{\sqcup\!\sqcup}(L) = L\hat{\otimes}1 + 1\hat{\otimes}L$ [23].
12. These operators are, in fact, the shifts of Harmonic Analysis and therefore defined as adjoints of multiplication, *i.e.*

$$\forall S \in \mathbb{C}\langle\langle x\rangle\rangle, \qquad \langle\delta_x^l(S) \mid w\rangle = \langle S \mid xw\rangle.$$

13. Thus, in this case, the operator δ_x^l has the same meaning as the operator $S \to S_x$ in [6], x^{-1} in the Theory of Languages and $\circ$ in [1, 23].

Using, once more, Lemma 1, one gets

PROPOSITION 2. *The family* $\{(\alpha_0 x_0)^* ⧢ (\alpha_1 x_1)^*\}_{\alpha_i\in\mathbb{C}}$ *is a* $(\mathbb{C}\langle X\rangle, ⧢, 1)$*-basis of* $\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$, *then we have a* $\mathbb{C}$*-basis* $\{w ⧢ (\alpha_0 x_0)^* ⧢ (\alpha_1 x_1)^*\}_{\substack{\alpha_i\in\mathbb{C}\\ w\in X^*}}$ *of*

$$\begin{aligned}\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle &= \mathbb{C}\langle X\rangle[\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle]\\ &= \mathbb{C}\langle X\rangle ⧢ SP_{\mathbb{C}}(X).\end{aligned}$$

PROOF. We will use a multi-parameter consequence of Lemma 1.

LEMMA 4. *Let* Z *be an alphabet, and* k *a field of characteristic zero. Then, the family* $\{e^{\alpha z}\}_{\substack{z\in Z\\ \alpha\in k}} \subset k[[Z]]$ *is linearly independent over* $k[Z]$.

This proves that, in the shuffle algebra the elements

$$\{(a_0 x_0)^* ⧢ (a_1 x_1)^*\}_{a_0,a_1\in\mathbb{C}^2}$$

are linearly independent over $\mathbb{C}\langle X\rangle \simeq \mathbb{C}[\mathscr{L}yn(X)]$ within $(\mathbb{C}\langle\langle X\rangle\rangle, ⧢, 1_{X^*})$. $\square$

Now $\mathrm{Li}_\bullet^{(2)}$ is well-defined and this morphism is not into from

$$\begin{aligned}\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle &=\\ \mathbb{C}\langle X\rangle[\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle] &= \mathbb{C}\langle X\rangle ⧢ SP_{\mathbb{C}}(X),\end{aligned}$$

to $\mathrm{Im}(\mathrm{Li}_\bullet^{(2)})$.

PROPOSITION 3. *Let* $\mathrm{Li}_\bullet^{(1)} : \mathbb{C}\langle X\rangle[x_0^*, x_1^*, (-x_0)^*] \to \mathscr{H}(\Omega)$ *then*

i. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$.

ii. $\ker(\mathrm{Li}_\bullet^{(1)})$ *is the ideal generated by* $x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}$.

PROOF. As $\mathbb{C}\langle X\rangle[x_0^*, x_1^*, (-x_0)^*]$ admits $\{(x_0^*)^{⧢ k} ⧢ (x_1^*)^{⧢ l}\}_{\substack{l\in\mathbb{N}\\ k\in\mathbb{Z}}}$ as a basis for its structure of $\mathbb{C}\langle X\rangle$-module, it suffices to remark

$$\mathrm{Li}^{(1)}_{(x_0^*)^{⧢ k} ⧢ (x_1^*)^{⧢ l}}(z) = z^k \times \frac{1}{(1-z)^l}$$

is a generating system of $\mathscr{C}$.

First of all, we recall the following lemma

LEMMA 5. *Let* M_1 *and* M_2 *be* K*-modules (*K *is a unitary ring). Suppose* $\phi : M_1 \to M_2$ *is a linear mapping. Let* $N \subset \ker(\phi)$ *be a submodule. If there is a system of generators in* M_1*, namely* $\{g_i\}_{i\in I}$*, such that*

1. *For any* $i \in I\setminus J$, $g_i \equiv \sum_{j\in J\subset I} c_i^j g_j [\mathrm{mod} N]$, $(c_i^j \in K; \forall j \in J)$ *;*
2. $\{\phi(g_j)\}_{j\in J}$ *is* K*-free in* M_2 *;*

then $N = \ker(\phi)$.

PROOF. Suppose $P \in \ker(\phi)$. Then $P \equiv \sum_{j\in J} p_j g_j [\mathrm{mod} N]$ with $\{p_j\}_{j\in J} \subset K$. Then $0 = \phi(P) = \sum_{j\in J} p_j \phi(g_j)$. From the fact that $\{\phi(g_J)\}_{J\in J}$ is $K-$ free on M_2, we obtain $p_j = 0$ for any $j \in J$. This means that $P \in N$. Thus $\ker(\phi) \subset N$. This implies that $N = \ker(\phi)$. $\square$

Let now $\mathscr{J}$ be the ideal generated by $x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}$. It is easily checked, from the following formulas, (for $l \geq 1$) [14]

$$w ⧢ x_0^* ⧢ (x_1^*)^{⧢ l} \equiv w ⧢ (x_1^*)^{⧢ l} - w ⧢ (x_1^*)^{⧢ l-1}\ [\mathscr{J}],$$

14. In figure 1, (w,l,k) codes the element $w ⧢ (x_0^*)^{⧢ l} ⧢ (x_1^*)^{⧢ k}$.

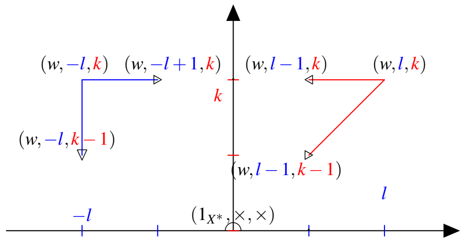

Figure 1: Rewriting $\mathrm{mod}\,\mathscr{J}$ **of** $\{w ⧢ (x_0^*)^{⧢ l} ⧢ (x_1^*)^{⧢ k}\}_{\substack{k\in\mathbb{N}, l\in\mathbb{Z}\\ w\in X^*}}$**.**

$$w ⧢ (-x_0)^* ⧢ (x_1^*)^{⧢ l} \equiv w ⧢ (-x_0)^* ⧢ (x_1^*)^{⧢ l-1} + w ⧢ (x_1^*)^{⧢ l} [\mathscr{J}],$$

that one can rewrite $[\mathrm{mod}\,\mathscr{J}]$ any monomial $w ⧢ (x_0^*)^{⧢ k} ⧢ (x_1^*)^{⧢ l}$ as a linar combination of such monomials with $kl = 0$. Then, applying lemma 5 to the map $\phi = \mathrm{Li}_\bullet^{(1)}$ and the modules

$$M_1 = \mathbb{C}\langle X\rangle[x_0^*, x_1^*, (-x_0)^*], \quad M_2 = \mathscr{H}(\Omega), \quad N = \mathscr{J},$$

the families and indices

$$\begin{aligned}\{g_i\} &= \{w ⧢ (x_1^*)^{⧢ n} ⧢ (x_0^*)^{⧢ m}\}_{(w,n,m)\in I},\\ I &= X^* \times \mathbb{N} \times \mathbb{Z},\\ J &= (X^* \times \mathbb{N} \times \{0\}) \sqcup (X^* \times \{0\} \times \mathbb{Z}),\end{aligned}$$

we have the second point of proposition 3.

Of course, we also have $(x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}) \subset \ker(\mathrm{Li}_\bullet^{(2)})$, but the converse is conjectural.

5. Applications on polylogarithms

Let us consider also the following morphisms $\Im$ and Θ of algebras $\mathbb{C}\langle X\rangle \to \mathrm{End}(\mathscr{C}\{\mathrm{Li}_w\})$ defined by

i. $\Im(w) = \mathrm{Id}$ and $\Theta(w) = \mathrm{Id}$, if $w = 1_{X^*}$.

ii. $\Im(w) = \Im(v)\iota_i$ and $\Theta(w) = \Theta(v)\theta_i$, if $w = vx_i, x_i \in X, v \in X^*$.

For any $n \geq 0$ and $u \in X^*, f, g \in \mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$, one has [5, 7]

$$\begin{aligned}\partial_z^n &= \sum_{w\in X^n} \mu\circ(\Theta\otimes\Theta)[\Delta_⧢(w)],\\ \Theta(u)(fg) &= \mu\circ(\Theta\otimes\Theta)[\Delta_⧢(u)]\circ(f\otimes g).\end{aligned}$$

By extension to complex coefficients, we obtain

$$\begin{aligned}\mathscr{H}_{\mathtt{conc}} &\cong (\mathbb{C}\langle\Theta(X)\rangle, \mathtt{conc}, \mathrm{Id}, \Delta_⧢, \varepsilon),\\ \mathscr{H}_⧢ &\cong (\mathbb{C}\langle\Im(X)\rangle, ⧢, \mathrm{Id}, \Delta_{\mathtt{conc}}, \varepsilon).\end{aligned}$$

Hence,

THEOREM 3 (DERIVATIONS AND AUTOMORPHISMS).

Let $P,Q\in\mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}[x_0^*,(-x_0)^*,x_1^*]\sqcup\!\sqcup\,\mathbb{C}\langle X\rangle$), $T\in\mathscr{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ (resp. $\mathscr{L}ie_{\mathbb{C}}\langle X\rangle$). Then $\Theta(T)$ is a derivation in $(\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1)$ (resp. $(\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1_\Omega)$) and $\exp(t\Theta(T))$ is then a one-parameter group of automorphisms of

$$(\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1_\Omega)\quad (\textit{resp.}\quad (\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1_\Omega)).$$

PROOF. Because $\mathrm{Li}_{P\sqcup\!\sqcup\, Q}=\mathrm{Li}_P\,\mathrm{Li}_Q,\Theta(T)\,\mathrm{Li}_{P\sqcup\!\sqcup\, Q}=\mathrm{Li}_{(P\sqcup\!\sqcup\, Q)\triangleleft T}$ and then $\Theta(T)(\mathrm{Li}_P\,\mathrm{Li}_Q)=\mathrm{Li}_{(P\sqcup\!\sqcup\, Q)\triangleleft T}=\mathrm{Li}_{(P\triangleleft T)\sqcup\!\sqcup\, Q+P\sqcup\!\sqcup\,(Q\triangleleft T)}=\mathrm{Li}_{(P\triangleleft T)\sqcup\!\sqcup\, Q}+\mathrm{Li}_{P\sqcup\!\sqcup\,(Q\triangleleft T)}=(\Theta(T)\,\mathrm{Li}_P)\,\mathrm{Li}_Q+\mathrm{Li}_P(\Theta(T)\,\mathrm{Li}_Q)$. $\square$

THEOREM 4 (EXTENSION OF $\mathrm{Li}_\bullet$).

The following map is surjective

$$\begin{array}{rcl}(\mathbb{C}[x_0^*]\sqcup\!\sqcup\,\mathbb{C}[(-x_0)^*]\sqcup\!\sqcup\,\mathbb{C}[x_1^*]\sqcup\!\sqcup\,\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*}) & \to & (\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*},\times,1),\\ T & \mapsto & \Im(T)1_\Omega.\end{array}$$

One has, for any $u\in Y^*$,

$$\mathrm{Li}^-_{y_{s_1}u}=\theta_0^{s_1}(\theta_0\iota_1)\,\mathrm{Li}^-_u=\theta_0^{s_1}(\lambda\,\mathrm{Li}^-_u)=\sum_{k_1=0}^{s_1}\binom{s_1}{k_1}(\theta_0^{k_1}\lambda)(\theta_0^{s_1-k_1}\,\mathrm{Li}^-_u).$$

Hence, successively [5],

$$\begin{aligned}\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}=&\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\\ &\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}(\theta_0^{k_1}\lambda)(\theta_0^{k_2}\lambda)\ldots(\theta_0^{k_r}\lambda),\end{aligned}$$

where

$$\theta_0^{k_i}\lambda(z)=\begin{cases}\lambda(z), & \text{if}\quad k_i=0,\\ \dfrac{1}{1-z}\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!\lambda^j(z), & \text{if}\quad k_i>0.\end{cases}$$

Hence,

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}=\mathrm{Li}_T=\Im(T)1_\Omega,$$

where T is the following exchangeable rational series

$$\begin{aligned}T\;=\;&\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\\ &\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}T_{k_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup\, T_{k_r},\\ T_{k_i}\;=\;&\begin{cases}(x_0+x_1)^*, & \text{if}\quad k_i=0,\\ x_1^*\sqcup\!\sqcup\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!((x_0+x_1)^*)^{\sqcup\!\sqcup\, j}, & \text{if}\quad k_i>0.\end{cases}\end{aligned}$$

Due to surjectivity of $\mathrm{Li}_\bullet$, from $\mathbb{C}[x_0^*]\sqcup\!\sqcup\,\mathbb{C}[(-x_0)^*]\sqcup\!\sqcup\,\mathbb{C}[x_1^*]\sqcup\!\sqcup\,\mathbb{C}\langle X\rangle$ to $\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}$, one also has

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}=\mathrm{Li}_R=\Im(R)1_\Omega,$$

where R is the following exchangeable rational series

$$\begin{aligned}R\;=\;&\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\\ &\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}R_{k_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup\, R_{k_r},\\ R_{k_i}\;=\;&\begin{cases}x_0^*\sqcup\!\sqcup\, x_1^*, & \text{if}\quad k_i=0,\\ x_1^*\sqcup\!\sqcup\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_0^*\sqcup\!\sqcup\, x_1^*)^{\sqcup\!\sqcup\, j}, & \text{if}\quad k_i>0,\end{cases}\end{aligned}$$

and again (see Example 1)

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}=\mathrm{Li}_F=\Im(F)1_\Omega,$$

where F is the following rational series on x_1

$$\begin{aligned}F\;=\;&\sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\\ &\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}F_{k_1}\sqcup\!\sqcup\ldots\sqcup\!\sqcup\, F_{k_r},\\ F_{k_i}\;=\;&\begin{cases}x_1^*-1_{X^*}, & \text{if}\quad k_i=0,\\ x_1^*\sqcup\!\sqcup\displaystyle\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_1^*-1_{X^*})^{\sqcup\!\sqcup\, j}, & \text{if}\quad k_i>0.\end{cases}\end{aligned}$$

Since $\Im(x_1^*)1_\Omega=1/(1-z)$ then this proves once again that [5, 7]

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}=\mathrm{Li}_T=\mathrm{L}_R=\mathrm{Li}_F\in\mathbb{C}[1/(1-z)]\subsetneq\mathscr{C}.$$

One can deduce finally that

COROLLARY 3.

$$\begin{aligned}\mathscr{C}\{\mathrm{Li}_w\}_{w\in X^*}\;&\supsetneq\;\mathbb{C}[1/(1-z)]\{\mathrm{Li}_w\}_{w\in X^*}\\ &=\;\mathrm{span}_{\mathbb{C}}\left\{\begin{array}{c}\displaystyle\sum_{n_1>\ldots>n_r>0}n_1^{s_1}\ldots n_r^{s_r}z^{n_1}\,|\\ (s_1,\ldots,s_r)\in\mathbb{Z}^r,r\in\mathbb{N}_+\end{array}\right\}.\end{aligned}$$

6. Conclusion

We have studied the structure of the algebra

$$\mathbb{C}\langle X\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle,$$

where $X=\{x_0,x_1\}$ is an alphabet. We have also considered the ways for denoting the polylogarithms. By the results on the algebra $\mathbb{C}\langle X\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$, we have given an extension of the polylogarithms and have obtained *polylogarithmic transseries*

$$\mathbb{C}\{z^\alpha(1-z)^\beta\,\mathrm{Li}_w\}_{\substack{w\in X^*\\ \alpha,\beta\in\mathbb{C}}}.$$

With this extension, we have constructed several shuffle bases of the algebra of polylogarithms. In the special case of the "Laurent subalgebra"

$$(\mathbb{C}\langle X\rangle,\sqcup\!\sqcup,1_{X^*})[x_0^*,(-x_0^*),x_1^*]\subset\mathbb{C}\langle X\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\!\sqcup\,\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle,$$

we have completely characterized the kernel of the polylogarithmic map $\mathrm{Li}_\bullet$, providing a rewriting process which terminates to a normal form.

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